

Magnetic Dipole Torque Theorem Proof With Green's Theorem

Raven Asher Raziel

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Prelude

Initially, after performing the 30 page proof of this theorem, I realized why not only textbooks do not give the gist of the proof, but also why the internet does not have it readily available. A few weeks later, I learned about Green's Theorem.

It is a welcome surprise whenever the generalized proof becomes shorter than the "simpler case". In this proof, the case of closed curves expands not only to those which are convex (due to the polar function being one-to-one), but simply those which are simple (does not self-intersect) and closed (begins and ends at the same position).

There is another method of utilizing Stoke's Theorem, but as Green's Theorem is a case of Stoke's Theorem, and certain intuitive invariance exist under a rotational linear map, it is not only intuitive but rigorous to say that the results for the xy plane apply to any plane.

I have spent a month deriving the original 30 page proof for polar coordinates. In a month or so, I might release a much more explicit version with Stoke's Theorem (which produces a normal vector with the magnitude of the area itself).

*For those who have not read the original paper: the textbook said that the torque is related to the area within ANY loop of wire, not just a rectangular or a circular one. It gave no indication of proof. Not only that, the internet did not posses any proof. Although these proofs may be out of reach at times for the student, yet useful for their level of learning, I believe that a proof should be posted **somewhere**. Math, unlike string physics supremacy - or Mormonism, is not a religion. Although it is made up, it is exact when declaring truths. And Science, is based on observation.*

Introduction to Torque on a Magnetic Loop

In many text books, the torque for a single loop of wire can be given by $\tau = \vec{u} \times \vec{B}$ where $\vec{u} = IA \hat{n}$. A is the area of the loop. The vector $\hat{n}$ is the unit vector given by the right hand rule if the fingers were curling in the direction of current and the thumb points up.

The equation for the total torque on a wire in a constant magnetic field is:

$$\tau = \oint_C \vec{r} \times (I \hat{l} \times \vec{B}) ds$$

Intuitively, the formula is adding the torque from every vector along the direction $\hat{l}(s)$ of infinitely small magnitude "ds" that composes a curve C

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Enough infinitesimal calculus.

According to the Magnetic Dipole Moment, the torque is related directly not the length of the wire but the area that it encloses. The proof for a rectangular loop and a circular loop are relatively simple. However, let us find a general proof.

The Proof

Let some simply closed curve C lie on the xy plane. Because it is a simple curve, there exists a parametrization:

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle, t \in [a, b]$$

$$\vec{r}(0) = \langle x_0, y_0, z_0 \rangle$$

$$\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle, t \in [a, b]$$

Thus, the magnetic torque is:

$$\tau = \oint_C \vec{r} \times (I \hat{l} \times \vec{B}) ds$$

The unit vector $\hat{l}$ points in the direction of current flow. This can be represented by the unit tangent vector of $\vec{r}$:

$$\tau = \oint_C \vec{r} \times \left(I \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \times \vec{B} \right) ds$$

When the Integral is expressed in terms of the parameter:

$$\begin{aligned} &= \oint_a^b \vec{r} \times \left(I \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \times \vec{B} \right) |\vec{r}'(t)| dt \\ &= \oint_a^b \vec{r} \times (I \vec{r}'(t) \times \vec{B}) dt \end{aligned}$$

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Important Lemmas

Relationship of Area with Circulation

Green's Theorem states that

$$\oint_C \vec{F} \cdot \vec{T} ds = \iint_R \text{curl } \vec{F} dA$$

If the 2 dimensional curl of F is equal to 1:

$$\iint_R 1 dA = \text{Area}$$

Consider these specific cases

$$\oint_C \langle 0, x \rangle \cdot \vec{T} ds = \oint_C \langle 0, x \rangle \cdot \langle x'(t), y'(t) \rangle dt = \oint_C x \frac{dy}{dt} dt = \oint_C x dy$$

$$\text{curl} \langle 0, x \rangle = \frac{\partial}{\partial x} [x] - \frac{\partial}{\partial y} [0] = 1$$

Therefore:

$$\oint_C \langle 0, x \rangle \cdot \vec{T} ds = \oint_C x dy = \iint_R 1 dA = \text{Area} = A$$

Consider the other trivial case:

$$\oint_C \langle y, 0 \rangle \cdot \vec{T} ds = \oint_C \langle y, 0 \rangle \cdot \langle x'(t), y'(t) \rangle dt = \oint_C y \frac{dx}{dt} dt = \oint_C y dx$$

$$\text{curl} \langle y, 0 \rangle = \frac{\partial}{\partial x} [0] - \frac{\partial}{\partial y} [y] = -1$$

$$\oint_C \langle y, 0 \rangle \cdot \vec{T} ds = \oint_C y dx = \iint_R -1 dA = -\text{Area} = -A \text{ (this is desirable)}$$

Alternate Form of the Theorem (to look out for)

$$\vec{u} \times \vec{B} = \tau$$

$$\vec{B} = \langle B_x, B_y, B_z \rangle$$

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Where I is the current and A is the area enclosed by the curve:

$$IA \hat{n} \times \vec{B} = \tau$$

On the xy plane, for a loop of wire with current flowing in the counterclockwise direction:

$$\hat{n} = \hat{k}$$

$$IA \hat{k} \times \langle B_x, B_y, B_z \rangle = IA \langle -B_y, B_x, 0 \rangle$$

Lagrange Expansion

There are many formulae named after the great Lagrange, let us invoke one of them:

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$$

(this can be easily proved, just expand the terms manually with basic vector algebra: good luck.)

$$\begin{aligned} \oint_a^b \vec{r} \times (I \vec{r}'(t) \times \vec{B}) dt &= I \oint_a^b (\vec{r}(t) \cdot \vec{B}) \vec{r}'(t) - (\vec{r}(t) \cdot \vec{r}'(t)) \vec{B} dt \\ &= I \oint_a^b (\vec{r}(t) \cdot \vec{B}) \vec{r}'(t) dt - I \oint_a^b (\vec{r}(t) \cdot \vec{r}'(t)) dt \vec{B} \end{aligned}$$

Cancellation of the simple term *with easier maths*

This section is left so that those who have not ventured too far into vector calculus can follow the proof confidently.

Consider the following:

$$I \oint_a^b (\vec{r}(t) \cdot \vec{r}'(t)) dt \vec{B}$$

$$\vec{r}(t) \cdot \vec{r}'(t) = x(t)x'(t) + y(t)y'(t) + z(t)z'(t)$$

$$I \oint_a^b (\vec{r}(t) \cdot \vec{r}'(t)) dt \vec{B} = I \oint_a^b x(t)x'(t) + y(t)y'(t) + z(t)z'(t) dt \vec{B}$$

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$$= I \vec{B} \left[\oint_a^b x(t) x'(t) + y(t) y'(t) + z(t) z'(t) dt \right]$$

$$= I \vec{B} \left[\oint_a^b x(t) x'(t) dt + \oint_a^b y(t) y'(t) dt + \oint_a^b z(t) z'(t) dt \right]$$

Because the starting and ending points of the curve are the same:

$$= I \vec{B} \left[\oint_{x_0}^{x_0} x(t) dx + \oint_{y_0}^{y_0} y(t) dy + \oint_{z_0}^{z_0} z(t) dz \right]$$

$$= I \vec{B} [0]$$

$$= 0$$

Therefore:

$$I \oint_a^b (\vec{r}(t) \cdot \vec{B}) \vec{r}'(t) dt - I \oint_a^b (\vec{r}(t) \cdot \vec{r}'(t)) dt \vec{B} = I \oint_a^b (\vec{r}(t) \cdot \vec{B}) \vec{r}'(t) dt - 0$$

$$\tau = \oint_C \vec{r} \times \left(I \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \times \vec{B} \right) ds = I \oint_a^b (\vec{r}(t) \cdot \vec{B}) \vec{r}'(t) dt$$

Cancellation of the simple term *elegantly*

$$I \oint_a^b (\vec{r}(t) \cdot \vec{r}'(t)) dt \vec{B}$$

The integral is simply the circulation of $\langle x, y, z \rangle$ on curve C.

If we prove that $\langle x, y, z \rangle$ is a conservative field, then by virtue of the closed path integral of a conservative vector field being zero, it too - in fact - unbelievably - shall be zero as well for any curve C including the simple closed ones we covet the most. As much as it is a radial field, and by mere inspection is irrotational - zero curl everywhere; however, for the sake of rigor, let us summon the pseudodeterminant (mnemonic abuse of notation).

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$$\nabla \times \langle x, y, z \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \langle (\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}), (\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}), (\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}) \rangle$$

$$= \langle (0-0), (0-0), (0-0) \rangle = \langle 0, 0, 0 \rangle = \vec{0}$$

It is conservative! Therefore, as the path integral is a closed loop, the integral reduces to zero.

Expansion of the Final Term

$$I \oint_a^b (\vec{r}(t) \cdot \vec{B}) \vec{r}'(t) dt$$

$$\vec{r}(t) \cdot \vec{B} = x(t)B_x + y(t)B_y$$

$$(\vec{r}(t) \cdot \vec{B}) \vec{r}'(t) = (x(t)B_x + y(t)B_y) x'(t)\hat{i} + (x(t)B_x + y(t)B_y) y'(t)\hat{j}$$

$$= x(t)x'(t)B_x\hat{i} + y(t)x'(t)B_y\hat{i} + x(t)y'(t)B_x\hat{j} + y(t)y'(t)B_y\hat{j}$$

$$I \oint_a^b x(t)x'(t)B_x\hat{i} + y(t)x'(t)B_y\hat{i} + x(t)y'(t)B_x\hat{j} + y(t)y'(t)B_y\hat{j} dt$$

$$= I \oint_a^b x(t)x'(t)B_x dt \hat{i} + I \oint_a^b y(t)x'(t)B_y dt \hat{i} + I \oint_a^b x(t)y'(t)B_x dt \hat{j} + I \oint_a^b y(t)y'(t)B_y dt \hat{j}$$

Just some rearranging

$$= I \oint_a^b x(t)x'(t)B_x dt \hat{i} + I \oint_a^b y(t)y'(t)B_y dt \hat{j}$$

$$+ I \oint_a^b y(t)x'(t)B_y dt \hat{i} + I \oint_a^b x(t)y'(t)B_x dt \hat{j}$$

Second Round of Canceling Terms

This was a lot less terrifying than the proof based off of polar functions.

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$$I \oint_a^b x(t) x'(t) B_x dt \hat{i} + I \oint_a^b y(t) y'(t) B_y dt \hat{j}$$

$$= I \oint_a^b x(t) x'(t) dt B_x \hat{i} + I \oint_a^b y(t) y'(t) dt B_y \hat{j}$$

$$= I \oint_{x_0}^{x_0} x(t) dx B_x \hat{i} + I \oint_{y_0}^{y_0} y(t) dy B_y \hat{j}$$

$$= 0 + 0 = 0$$

Therefore

$$\begin{aligned} &= I \oint_a^b x(t) x'(t) B_x dt \hat{i} + I \oint_a^b y(t) y'(t) B_y dt \hat{j} \\ &+ I \oint_a^b y(t) x'(t) B_y dt \hat{i} + I \oint_a^b x(t) y'(t) B_x dt \hat{j} \\ &= I \oint_a^b y(t) x'(t) B_y dt \hat{i} + I \oint_a^b x(t) y'(t) B_x dt \hat{j} + 0 \end{aligned}$$

Emergence of Area

Even though we are in $\mathbb{R}^3$, the integral lies on the xy plane, which is $\mathbb{R}^2$; therefore, Green's Theorem Applies. A much more complicated, yet more straightforward way to prove this is with Stokes theorem where the area.

And in order to generalize it for any plane, it must be proven that the magnitude of the curl is invariant under a rotation matrix and that the direction of the curl vector is rotated in the same manner as the unit vectors. (Coming soon this January).

$$\tau = I \oint_a^b y(t) x'(t) B_y dt \hat{i} + I \oint_a^b x(t) y'(t) B_x dt \hat{j}$$

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$$= I \oint_a^b y(t) x'(t) dt B_y \hat{i} + I \oint_a^b x(t) y'(t) dt B_x \hat{j}$$

$$= I \oint_a^b y(t) dx B_y \hat{i} + I \oint_a^b x(t) dy B_x \hat{j}$$

The integrals satisfy the area lemmas proven earlier.

$$= I \iint_R -1 dA B_y \hat{i} + I \iint_R 1 dA B_x \hat{j}$$

$$= -IAB_y \hat{i} + IAB_x \hat{j}$$

This vector sum is the same in $\mathbb{R}^2$ and in $\mathbb{R}^3$ which means that they are equivalent.

$$IA \hat{k} \times \langle B_x, B_y, B_z \rangle = IA \langle -B_y, B_x, 0 \rangle$$

$$IA \hat{n} \times \vec{B} = \tau$$

Equal Torque Everywhere Theorem

A curve of the same shape can be evaluated with a different center; however, since the area within both curves are the same, it implies that the torque within a closed loop of wire is the same about every point within; furthermore, as it is not required for the point to be within the curve (regarding the $\mathbf{r}$ function), the torque of a loop of wire that is mounted on a pivot outside its area is the same: assuming a constant magnetic field. (Intuitively, one end of the wire will tend towards the counterclockwise while the other will tend towards counterclockwise. Even though the lever arm is longer, the net effect will be the same).

Conclusion

The pursuit of mathematical and scientific knowledge can often become esoteric. At times, the textbook will wave its hand. Although it is the instinct to avoid pain, great insight can be gained by exploring these windows. Should one reach the edge of a branch of mathematics, it will be offered to them to extend it into the void. By practicing extending the branch into the void, one can be well prepared for this.

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I learned that whenever a problem is stuck, it is best to create a rhythm of attempting it and then doing something else. The original form of this proof took a month to complete, and this form took only an afternoon. But it was the heavy lifting gained from the previous iteration that gave me the strength.

When a problem is too difficult, insights could be gained by considering simpler cases such as when I proved the circular loop, and then extended it into general closed polar equations.

Finally, sometimes things will not be simple. As my Calculus 3 professor, Anthony Vance says, "Cope", and "Embrace the Complexity". At times, without that complexity, certain terms would simple not cancel out, rendering the problem more difficult.